ISSN 1464-8997 (on line) 1464-8989 (printed)

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part I, section 13, pages 113–116

13. Abelian extensions of absolutely unramified complete discrete valuation fields

Masato Kurihara

In this section we discuss results of [K]. We assume that p is an odd prime and K is an absolutely unramified complete discrete valuation field of mixed characteristics $(0,p)$, so p is a prime element of the valuation ring $\mathcal{O}_K$. We denote by F the residue field of K.

13.1. The Milnor K-groups and differential forms

For $q > 0$ we consider the Milnor K-group $K_q(K)$, and its p-adic completion $\widehat{K}_q(K)$ as in section 9. Let $U_1\widehat{K}_q(K)$ be the subgroup generated by $\{1+p\mathcal{O}_K, K^*, \dots, K^*\}$. Then we have:

Theorem. *Let K be as above. Then the exponential map $\exp_p$ for the element p, defined in section 9, induces an isomorphism*

$$\exp_p\colon \widehat{\Omega}^{q-1}_{\mathcal{O}_K}/pd\widehat{\Omega}^{q-2}_{\mathcal{O}_K} \xrightarrow{\sim} U_1\widehat{K}_q(K).$$

The group $\widehat{K}_q(K)$ carries arithmetic information of K, and the essential part of $\widehat{K}_q(K)$ is $U_1\widehat{K}_q(K)$. Since the left hand side $\widehat{\Omega}^{q-1}_{\mathcal{O}_K}/pd\widehat{\Omega}^{q-2}_{O_K}$ can be described explicitly (for example, if F has a finite p-base I, $\widehat{\Omega}^1_{\mathcal{O}_K}$ is a free $\mathcal{O}_K$-module generated by $\{dt_i\}$ where $\{t_i\}$ are a lifting of elements of I), we know the structure of $U_1\widehat{K}_q(K)$ completely from the theorem.

In particular, for subquotients of $\widehat{K}_q(K)$ we have:

Corollary. *The map $\rho_m\colon \Omega^{q-1}_F \oplus \Omega^{q-2}_F \longrightarrow \mathrm{gr}_m K_q(K)$ defined in section 4 induces an isomorphism*

$$\Omega^{q-1}_F/B_{m-1}\Omega^{q-1}_F \xrightarrow{\sim} \mathrm{gr}_m K_q(K)$$

where $B_{m-1}\Omega_F^{q-1}$ *is the subgroup of* Ω_F^{q-1} *generated by the elements* $a^{p^j} d\log a \wedge d\log b_1 \wedge \dots \wedge d\log b_{q-2}$ *with* $0 \leqslant j \leqslant m-1$ *and* $a,\ b_i \in F^*$.

13.2. Cyclic p-extensions of K

As in section 12, using some class field theoretic argument we get arithmetic information from the structure of the Milnor K-groups.

Theorem. *Let* $W_n(F)$ *be the ring of Witt vectors of length* n *over* F. *Then there exists a homomorphism*

$$\Phi_n\colon H^1(K,\mathbb{Z}/p^n) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\overline{K}/K),\mathbb{Z}/p^n) \longrightarrow W_n(F)$$

for any $n \geqslant 1$ *such that:*

(1) *The sequence*

$$0 \to H^1(K_{\mathrm{ur}}/K,\mathbb{Z}/p^n) \to H^1(K,\mathbb{Z}/p^n) \xrightarrow{\Phi_n} W_n(F) \to 0$$

is exact where K_{ur} *is the maximal unramified extension of* K.

(2) *The diagram*

$$\begin{array}{ccc} H^1(K,\mathbb{Z}/p^{n+1}) & \xrightarrow{\ p\ } & H^1(K,\mathbb{Z}/p^n) \\ \big\downarrow \Phi_{n+1} & & \big\downarrow \Phi_n \\ W_{n+1}(F) & \xrightarrow{\ \mathbf{F}\ } & W_n(F) \end{array}$$

is commutative where $\mathbf{F}$ *is the Frobenius map.*

(3) *The diagram*

$$\begin{array}{ccc} H^1(K,\mathbb{Z}/p^{n}) & \longrightarrow & H^1(K,\mathbb{Z}/p^{n+1}) \\ \big\downarrow \Phi_{n} & & \big\downarrow \Phi_{n+1} \\ W_{n}(F) & \xrightarrow{\ \mathbf{V}\ } & W_{n+1}(F) \end{array}$$

is commutative where $\mathbf{V}((a_0,\dots,a_{n-1})) = (0,a_0,\dots,a_{n-1})$ *is the Verschiebung map.*

(4) *Let* E *be the fraction field of the completion of the localization* $O_K[T]_{(p)}$ *(so the residue field of* E *is* $F(T)$*). Let*

$$\lambda\colon W_n(F)\times W_n(F(T)) \xrightarrow{\rho} {}_{p^n}\mathrm{Br}(F(T)) \oplus H^1(F(T),\mathbb{Z}/p^n)$$

be the map defined by $\lambda(w,w') = (i_2(p^{n-1}wdw'), i_1(ww'))$ *where* ${}_{p^n}\mathrm{Br}(F(T))$ *is the* p^n*-torsion of the Brauer group of* $F(T)$, *and we consider* $p^{n-1}wdw'$ *as an element of* $W_n\Omega^1_{F(T)}$ *(*$W_n\Omega^{\cdot}_{F(T)}$ *is the de Rham Witt complex). Let*

$$i_1\colon W_n(F(T)) \longrightarrow H^1(F(T),\mathbb{Z}/p^n)$$

be the map defined by Artin–Schreier–Witt theory, and let

$$i_2 \colon W_n\Omega^1_{F(T)} \longrightarrow {}_{p^n}\operatorname{Br}(F(T))$$

be the map obtained by taking Galois cohomology from an exact sequence

$$0 \longrightarrow (F(T)^{\mathrm{sep}})^*/((F(T)^{\mathrm{sep}})^*)^{p^n} \longrightarrow W_n\Omega^1_{F(T)^{\mathrm{sep}}} \longrightarrow W_n\Omega^1_{F(T)^{\mathrm{sep}}} \longrightarrow 0.$$

Then we have a commutative diagram

$$\begin{array}{ccccc} H^1(K,\mathbb{Z}/p^n) & \times E^*/(E^*)^{p^n} & \xrightarrow{\ \cup\ } & \operatorname{Br}(E) \\ \Phi_n\big\downarrow & \big\uparrow\psi_n & & \big\uparrow i \\ W_n(F) & \times W_n(F(T)) & \xrightarrow{\ \lambda\ } & {}_{p^n}\operatorname{Br}(F(T)) \oplus H^1(F(T),\mathbb{Z}/p^n) \end{array}$$

where i is the map in subsection 5.1, and

$$\psi_n((a_0,\ \dots, a_{n-1})) = \exp\Big(\sum_{i=0}^{n-1}\sum_{j=1}^{n-i} p^{i+j}\,\widetilde{a}_i^{\,p^{n-i-j}}\Big)$$

($\widetilde{a}_i$ is a lifting of a_i to $\mathcal{O}_K$).

(5) *Suppose that $n = 1$ and F is separably closed. Then we have an isomorphism*

$$\Phi_1 \colon H^1(K,\mathbb{Z}/p) \simeq F.$$

Suppose that $\Phi_1(\chi) = a$. Then the extension L/K which corresponds to the character χ can be described as follows. Let $\widetilde{a}$ be a lifting of a to $\mathcal{O}_K$. Then $L = K(x)$ where x is a solution of the equation

$$X^p - X = \widetilde{a}/p.$$

The property (4) characterizes Φ_n.

Corollary (Miki)**.** *Let $L = K(x)$ where $x^p - x = a/p$ with some $a \in \mathcal{O}_K$. L is contained in a cyclic extension of K of degree p^n if and only if*

$$a \bmod p \in F^{p^{n-1}}.$$

This follows from parts (2) and (5) of the theorem. More generally:

Corollary. *Let χ be a character corresponding to the extension L/K of degree p^n, and $\Phi_n(\chi) = (a_0,\ \dots, a_{n-1})$. Then for $m > n$, L is contained in a cyclic extension of K of degree p^m if and only if $a_i \in F^{p^{m-n}}$ for all i such that $0 \leqslant i \leqslant n-1$.*

Remarks.

(1) Fesenko gave a new and simple proof of this theorem from his general theory on totally ramified extensions (cf. subsection 16.4).

(2) For any $q > 0$ we can construct a homomorphism

$$\Phi_n\colon H^q(K, \mathbb{Z}/p^n(q-1)) \longrightarrow W_n\Omega_F^{q-1}$$

by the same method. By using this homomorphism, we can study the Brauer group of K, for example.

Problems.

(1) Let $\chi_{\text{Ж}}$ be the character of the extension constructed in 14.1. Calculate $\Phi_n(\chi_{\text{Ж}})$.

(2) Assume that F is separably closed. Then we have an isomorphism

$$\Phi_n\colon H^1(K, \mathbb{Z}/p^n) \simeq W_n(F).$$

This isomorphism is reminiscent of the isomorphism of Artin–Schreier–Witt theory. For $w = (a_0, \ldots, a_{n-1}) \in W_n(F)$, can one give an explicit equation of the corresponding extension L/K using $a_0, \ldots, a_{n-1}$ for $n \geqslant 2$ (where L/K corresponds to the character χ such that $\Phi_n(\chi) = w$)?

References

[K] M. Kurihara, Abelian extensions of an absolutely unramified local field with general residue field, Invent. math., 93 (1988), 451–480.

Department of Mathematics Tokyo Metropolitan University
Minami-Osawa 1-1, Hachioji, Tokyo 192-03, Japan
E-mail: m-kuri@comp.metro-u.ac.jp